Natural Science - Oral Presentation
Markov Chains and Snakes and Ladders

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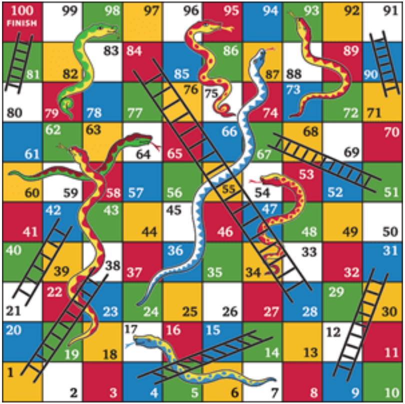
Introduction

Snakes and Ladders



Introduction Cont.

Rules of Snakes and Ladders

- Start at space 0
- Roll a 6 sided die and move that many spaces forward
- Go up ladders
- Go down snakes
- Land exactly at 100 or "bounce" back

Statement of Project

Objective

The project aims to use **Markov Chain** models and its theorems and apply it to Snakes and Ladders to make it more competitive and fair.

Statement of Project Cont.

Definition of Fair and Competitive

I aim to make a board where the first roll is insignificant, where the expected number of rolls needed to go from space 1 to space 100 is about the same as going from space 6 to space 100.

Standard Deviation

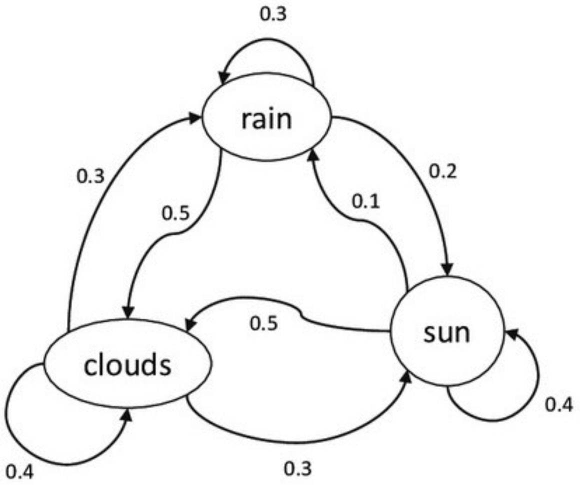
A low standard deviation of the expected number moves to get the end of the board starting from space 1,2,3,4,5, and 6, means a more a fair game of Snakes and Ladders

Definitions and Background

Definition

Markov Chains: A Markov chain or Markov process is a stochastic model describing a sequence of possible events in which the probability of each event depends only on the state attained in the previous event.

Definitions and Background cont.



Definitions and Background cont.

	clouds	rain	sun
clouds	0.4	0.3	0.3
rain	0.5	0.3	0.2
sun	0.5	0.1	0.4

Definition

The *Transition Matrix* for a Markov chain is the matrix P with entries p_{ij}. The initial probability vector is the vector $\pi_0 = p_j^{(0)} = Pr[f_0 = s_j]$

Definitions and Background cont.

Different Kind of States

The minimal elements of the partial ordering of equivalence classes are called ergodic sets. The remaining elements are called transient sets. The element of a transient set are called transient states. The elements of an ergodic set are called ergodic (or non-transient) states.

Definitions and Background cont.

Transition Matrix:

$$\begin{bmatrix} I & 0 \\ R & Q \end{bmatrix}.$$

Where I is the identity matrix, 0 is the matrix of all zeroes, R represents the transition from transient to ergodic sets, and Q represents the process that stays in the transient states.

Definitions and Background cont.

Fundamental Matrix:

$$N = (I - Q)^{-1}$$

Theorem

Each value $N_{i,j}$ in N represents the expected number of turns the piece stays in j when it starts the game at i.

Proof of Fundamental Matrix

Some Theory of the Fundamental Matrix

Theorem 3.2

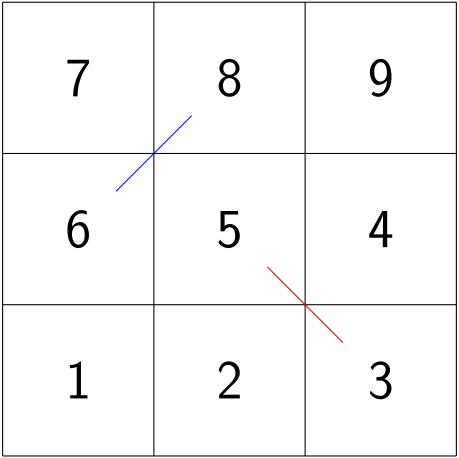
For any absorbing Markov Chain. $I - Q$ has an inverse and

$$(I - Q)^{-1} = I + Q + Q^2 \ldots = \sum_{k-0}^{\infty} Q^k$$

Theorem 3.4

$$M_i[n_j] = N \text{ where } s_i, s_j \in T$$

Sample 3x3 Board



Sample 3x3 Board Rules

Rules of 3x3 board

- Start at space 0
- Roll a 4 sided die and move that many spaces forward
- Go up ladders (blue lines)
- Go down snakes (red lines)
- Land exactly at 9 or “bounce” back

Transition Matrix without snakes and ladders

$$\begin{bmatrix} 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{2}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{2}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

There is a diagonal of $\frac{1}{4}$ until the end

Transition Matrix With Snakes and Ladders

$$\begin{bmatrix} 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} & \frac{2}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2}{4} & \frac{1}{4} & 0 & 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & \frac{1}{4} & \frac{2}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{2}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{2}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Notice some patterns in this Transition Matrix

Expected Value

Using a Python code to find the Fundamental Matrix, we then get the resulting vector of Expected Values with Snakes and ladders to be:

$$\begin{bmatrix} 7.09659091 \\ 6.84090909 \\ 6.27272727 \\ 5.81818182 \\ 5.45454545 \\ 1 \\ 1 \\ 4 \\ 4 \end{bmatrix}$$

Expected Value cont.

We remove all of the "impossible" states, thus getting

$$\begin{bmatrix} 7.09659091 \\ 6.84090909 \\ 6.27272727 \\ 5.81818182 \\ 5.45454545 \\ 4 \\ 4 \end{bmatrix}$$

With/Without Snakes and Ladders

Expected value:

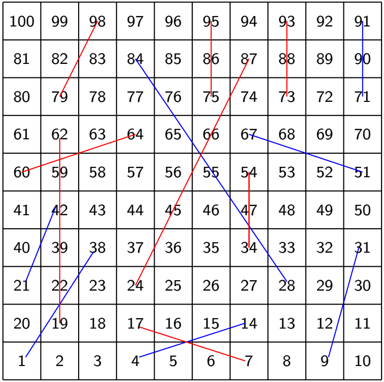
$$\text{With: } \begin{bmatrix} 7.09659091 \\ 6.84090909 \\ 6.27272727 \\ 5.81818182 \\ 5.45454545 \\ 4 \\ 4 \end{bmatrix} \quad \text{Without: } \begin{bmatrix} 6.44140625 \\ 5.953125 \\ 5.5625 \\ 5.25 \\ 5 \\ 4 \\ 4 \\ 4 \\ 4 \end{bmatrix}$$

Standard Deviation

Standard Deviation of the Expected Value starting from space 1,2,3, and 4:

SD without snakes and ladders: 0.35636905220177
SD with snakes and ladders: 0.51835498562612

Bigger 10x10 board



Ladders:$(1, 38), (4, 14), (9, 31), (21, 42), (28, 84), (51, 67), (71, 91)$

Snakes:$(17, 7), (54, 34), (62, 19), (64, 60), (87, 24), (93, 73), (95, 75), (98, 79)$

Expected Value of 10x10 Board

Expected Value:

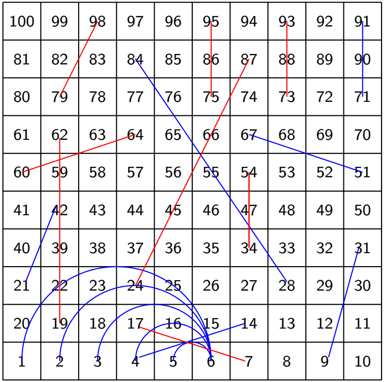
[78.01729741, 83.21632029, 82.75045955, 80.6725099, 82.78735372, 82.59279913].

Standard Deviation;

1.8255461673323

Revision to the 10x10 Board

Impossible/Unrealistic Revision:

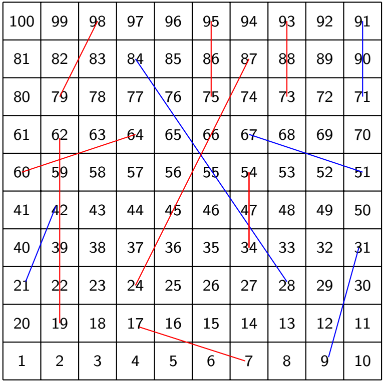


Ladders: $(1,6),(2,6),(3,6),(4,6),(5,6),(9,31),(21,42),(28,84),(51,67),(71,91)$

Snakes: $(17,7),(54,34),(62,19),(64,60),(87,24),(93,73),(95,75),(98,79)$

Revisions cont.

Revision 1:



Ladders:$[(9,31),(21,42),(28,84),(51,67),(71,91)]$

Snakes :$[(17,7),(54,34),(62,19),(64,60),(87,24),(93,73),(95,75),(98,79)]$

Expected Value and Standard Deviation

Expected Value

[83.91201371, 83.65690523, 83.12810378, 82.93837528, 82.78735372, 82.59279913].

Standard Deviation

.46951075134394

Thank you!